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Transient analytical solution to heat conduction in multi-dimensional composite cylinder slab

X. Lu [a,*], P. Tervola [b], M. Viljanen [a]

[a] Laboratory of Structural Engineering and Building Physics, Department of Civil and Environmental Engineering, Helsinki University of Technology, PL 2100, FIN-02015 HUT, Espoo, Finland
[b] Andritz Group, Tammasaarenkatu 1, FIN-00180 Helsinki, Finland

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Abstract

The analytical solution for the problem of transient heat conduction in multi-dimensional composite cylinder slab is developed for a time-dependent boundary condition. For such problems, numerical programs are needed to obtain eigenvalues and residues in most of the published papers. The numerical schemes may become unstable due to the existence of imaginary eigenvalues in multi-dimensional cases. In this paper, the proposed analytical method involves no numerical complications. By a novel application of the methods of the Laplace transform and separation of variables together with variable transformations, the residue calculation is avoided. The developed analytical method is powerful which represents extension of the analytical approach derived for the heat conduction problem in Cartesian coordinates. A closed form solution is provided. Calculation examples show that the analytical solutions predict good agreement with the numerical results.

Keywords: Multi-dimension; Composite cylinder slab; Heat conduction; Analytical method; Closed form solution

1. Introduction

The use of composite cylinders has been tremendous in many engineering fields such as aerospace, automobile, chemical and energy, civil and infrastructure, sports and recreation, and even biomedical engineering. As a result, a detailed knowledge of temperature distributions and heat fluxes in composite cylinders is needed in heat conduction problems. Numerical methods are a common method for such problems, however, analytical approaches can provide greater insight into the physical processes and can be used to validate numerical models. Unfortunately, analytical solutions exist only for relatively simple cases wherein: (1) the slab is homogeneous and (2) the boundary conditions are not complicated. To deal with practical problems of composite cylinders with general boundary conditions, most of the analytical methods are limited by high computational cost involving numerical iterations for eignevalues and residues. Such eigenvalue and residue computations are often not fully automatic and, consequently, are inherently time-consuming [1].

For multi-dimensional heat conduction problems in a composite cylinder slab, the commonly applied techniques are Green functions, orthogonal expansions and the Laplace transform [2]. The first two techniques inherit associated eigenvalue problems. In a single layer slab in one-dimensional case, an eigenfunction often links the space and time variables when applying separation of variables. However, in a multi-layer slab in one-dimensional case, eigenfunctions may be also yielded from the boundary conditions presented in the contacted layers. Hence, eigenvalue problems may exist even for steady-state heat conduction problems in one-dimensional geometry. The statement is true for a multi-dimensional slab also (see [3]

* Corresponding author. Tel.: +358 9 4515305; fax: +358 9 4513724.
E-mail address: xiaoshu@cc.hut.fi (X. Lu).

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Nomenclature

b	resultant coefficient defined in Eq. (3.15)	T	temperature
h	intermediate variable defined in Eq. (3.22a)	U	homogenised temperature $= T - T_\infty$
j	layer identifier	V	constructed new variable defined in Eq. (3.19)
k	diffusivity	X	variable-separated temperature $U = XR$
L, l	thickness	x	space coordinate
M	resultant function defined in Eq. (3.24a)		
m	index number	*Greek symbols*	
N	resultant function defined in Eq. (3.24b)	α	convective and radiative heat transfer coefficient
n	layer number	φ	phase
q	intermediate variable defined in Eq. (3.22a)	λ	thermal conductivity
R	variable-separated temperature $U = XR$	ω	period
r	space coordinate	ξ	intermediate variable defined in Eq. (3.22a)
t	time		

as an example). Concerning the third technique of the Laplace transform, residue computations are often needed. A detailed literature review of these methods can be found in [4]. As an example, an analytical solution for the problem of transient heat conduction in two-layer cylinder slab was proposed in [5]. Several groups of eigenvalues were needed to compute. The dependence of the eigenfunctions of the eigenvalues was 'very awkwardly', the quotation mark presenting the quoted text in [5]. Moreover, the numerical searching program may become unstable due to the existence of the imaginary eigenvalues [6].

Recently, Lu et al. developed an analytical method for the multi-dimensional transient heat problem in a composite slab subject to a time-dependent boundary condition [7]. An approximated solution is obtained. The main contribution of the work is a novel application of the methods of the Laplace transform and separation of variables together with variable transformations, a numerical work concerning eigenvalue and residue search is avoided [7].

The objective of this work is to extend the result in Cartesian coordinates [7] to cylindrical coordinates. The configuration of the problem dealt in this paper is similar as that considered in [5]. Only one group of eigenvalues is needed to find. The eigenvalues represent the roots of a simple Bessel function, which can be easily obtained from many standard textbooks. Hence, no numerical work is necessitated. Even in an extreme case when numerical searching of these eigenvalues is needed, an instability risk in numerical search due to imaginary eigenvalues does not exist. Such instability risk problem is very common in solving multi-dimensional multi-layer heat conduction problems [6].

2. Mathematical model

2.1. Problem specification and model equations

Let an n-layer composite cylinder be in cylindrical coordinates in x- and r-directions as illustrated in Fig. 1. The layers are in x-direction and characterised by constant conductivity, diffusivity and thickness which are denoted as λ_j, k_j and l_j, $j = 1, \ldots, n$. The cylinders have a common radius r_0. An ideal contact between layers is assumed.

Denote $L_0 = l_0 = 0$ and $L_j = l_0 + \cdots + l_j$, $j = 1, \ldots, n$. So the lengths of contact layers in x-direction are L_0, $L_1, \ldots, L_n$. The general heat conduction equation in terms of temperature $T_j(t, r, x)$ in cylindrical coordinates is

$$k_j\left(\frac{\partial^2 T_j}{\partial r^2} + \frac{1}{r}\frac{\partial T_j}{\partial r}\right) + k_j\frac{\partial^2 T_j}{\partial x^2} = \frac{\partial T_j}{\partial t},$$
$$0 < r < r_0,\ \ x \in [L_{j-1}, L_j],\ \ j = 1, \ldots, n, \tag{2.1a}$$

with boundary conditions

$$\lambda_1\frac{\partial T_1}{\partial x}(t, r, L_0) = -\alpha_+(T_1(t, r, L_0) - T_\infty(t)),$$
$$0 < r < r_0, \tag{2.1b}$$

$$T_j(t, r, L_j) = T_{j+1}(t, r, L_j),$$
$$0 < r < r_0,\ \ j = 1, \ldots, n-1, \tag{2.1c}$$

$$-\lambda_j\frac{\partial T_j}{\partial x}(t, r, L_j) = -\lambda_{j+1}\frac{\partial T_{j+1}}{\partial x}(t, r, L_j),$$
$$0 < r < r_0,\ \ j = 1, \ldots, n-1, \tag{2.1d}$$

$$-\lambda_n\frac{\partial T_n}{\partial x}(t, r, L_n) = -\alpha_-(T_\infty(t) - T_n(t, r, L_n)),$$
$$0 < r < r_0, \tag{2.1e}$$

$$-\lambda_j\frac{\partial T_j}{\partial r}(t, r_0, x) = -\alpha_{rj}(T_j(t, r_0, x) - T_\infty(t)),$$
$$x \in [L_{j-1}, L_j],\ \ j = 1, \ldots, n, \tag{2.1f}$$

and the initial value

$$T_j(0, r, x) = 0, \quad 0 < r < r_0,\ \ x \in [L_{j-1}, L_j],\ \ j = 1, \ldots, n. \tag{2.1g}$$

Without losing generality, it is assumed that the initial temperature is zero. The surface heat transfer coefficients are denoted as α_+, α_- and α_{rj}, $j = 1, \ldots, n$. The boundary temperature is presented as time-dependent $T_\infty(t)$.

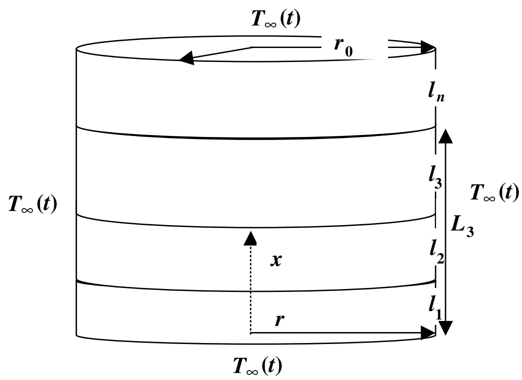

Fig. 1. Schematic of the composite cylinder slab.

2.2. Further statement of the boundary conditions

For calculational convenience, the boundary temperature is assumed as a simple periodic function $T_\infty(t) = \cos(\omega t + \varphi)$. Furthermore, a solution will be given according to the complex form of the boundary temperature, namely

$$T_\infty(t) = e^{i\omega t + i\varphi}. \tag{2.2}$$

Hence the solution of Eq. (2.1) will be the real part of the sought-after solution. If there is no danger of confusion, we shall keep the same notations. For more general boundary temperatures, solutions will be derived later.

In general study contexts, it has been agreed that the boundary condition of the third kind can produce mathematical incompatibilities in the direction parallel to the layers [7]. Hence, only the first and the second kind boundary conditions in x-direction are considered here, which assumes that α_{rj} are either zero or ∞ (first and second kinds). As a result, two types of boundary conditions in x-direction are considered:

$$r\text{-boundary-1}: \quad \alpha_{rj} = \infty, \quad j = 1, \ldots, n, \tag{2.3a}$$

$$r\text{-boundary-2}: \quad \alpha_{rj} = 0, \quad j = 1, \ldots, n. \tag{2.3b}$$

The solution for r-boundary-2 problem can be approximated as one-dimensional problem in Cartesian coordinates which has been studied earlier [4]. Therefore, we shall focus on developing the solution method for r-boundary-1 case.

3. Solution method

3.1. Homogenising the equations

For $\alpha_{rj} = \infty$, Eq. (2.1f) reads

$$T_j(t, r_0, x) = T_\infty(t), \quad x \in [L_{j-1}, L_j], \ j = 1, \ldots, n. \tag{3.1a}$$

For any jth layer, we introduce the following new variable in order to homogenise the boundary condition:

$$U_j = T_j - T_\infty(t). \tag{3.1b}$$

Eq. (2.1) is then re-written as

$$k_j\left(\frac{\partial^2 U_j}{\partial r^2} + \frac{1}{r}\frac{\partial U_j}{\partial r}\right) + k_j\frac{\partial^2 U_j}{\partial x^2} = \frac{\partial U_j}{\partial t} + T'_\infty(t),$$
$$0 < r < r_0, \ x \in [L_{j-1}, L_j], \ j = 1, \ldots, n, \tag{3.2a}$$

with boundary and initial conditions

$$-\lambda_1\frac{\partial U_1}{\partial x}(t, r, L_0) = -\alpha_+ U_1(t, r, L_0), \quad 0 < r < r_0, \tag{3.2b}$$

$$U_j(t, r, L_j) = U_{j+1}(t, r, L_j),$$
$$0 < r < r_0, \ j = 1, \ldots, n-1, \tag{3.2c}$$

$$-\lambda_j\frac{\partial U_j}{\partial x}(t, r, L_j) = -\lambda_{j+1}\frac{\partial U_{j+1}}{\partial x}(t, r, L_j),$$
$$0 < r < r_0, \ j = 1, \ldots, n-1, \tag{3.2d}$$

$$-\lambda_n\frac{\partial U_n}{\partial x}(t, r, L_n) = \alpha_- U_n(t, r, L_n), \quad 0 < r < r_0, \tag{3.2e}$$

$$U_j(0, r_0, x) = 0, \quad x \in [L_{j-1}, L_j], \ j = 1, \ldots, n. \tag{3.2f}$$

$$U_j(0, r, x) = -T_\infty(0),$$
$$0 < r < r_0, \ x \in [L_{j-1}, L_j], \ j = 1, \ldots, n. \tag{3.2g}$$

3.2. Separating the variables

As Eq. (3.2a) is nonhomogeneous, we shall adopt a novel technique of separation of variables by assuming that

$$U_j(t, r, x) = X_j(t, x)R_j(r), \tag{3.3}$$

where $R_j(r)$ is a variable-separated function which satisfies the homogeneous form of Eq. (3.2a). By substituting $R_j(r)$ into the homogeneous form of Eq. (3.2a) we get

$$\text{function of } t \text{ and } x = \frac{k_j\left(\frac{d^2R_j}{dr^2} + \frac{1}{r}\frac{dR_j}{dr}\right)}{R_j}. \tag{3.4}$$

Setting each side of the above equation as $-k_j\mu_j^2$ gives

$$k_j\left(\frac{d^2R_j}{dr^2} + \frac{1}{r}\frac{dR_j}{dr}\right) + k_j\mu_j^2 R_j = 0, \tag{3.5}$$

whose general solution is

$$R_{jm} = A_{jm}J_0(\mu_{jm}r), \quad m = 1, \ldots, \infty, \tag{3.6}$$

where J_0 is the Bessel function of the first kind of order zero and A_{jm} are determined by the boundary conditions. To satisfy Eq. (3.2f) $R_{jm}(r_0) = 0$, we get the following eigenfunction:

$$J_0(\mu_{jm}r_0) = 0. \tag{3.7}$$

The roots of J_0 have been studied extensively in many engineering problems. The eigenvalues are tabulated in many standard books. We can list the first five values:

$$\mu_{jm}r_0 = 2.4048, 5.5201, 8.6537, 11.7915, 14.9303. \tag{3.8}$$

The coefficient μ_{jm} is independent of layer j and can be written as

$$\mu_m = \frac{2.4048}{r_0}, \frac{5.5201}{r_0}, \ldots, \quad m = 1, \ldots, \infty. \tag{3.9}$$

Hence R_m in Eq. (3.6) can be obtained and the solution U_j in Eq. (3.3) is presented as

$$U_j(t,r,x) = \sum_{m=1}^{\infty} X_{jm}(t,x) R_m(r) = \sum_{m=1}^{\infty} X_{jm}(t,x) J_0(\mu_m r). \tag{3.10}$$

Note that the coefficient A_{jm} in Eq. (3.6) is embedded in X_{jm}.

3.3. Some properties of r-variable function R_m

We shall omit writing $m = 1, \ldots, \infty$, $j = 1, \ldots, n$, etc. Before deriving the equations for X_{jm}, let us recall some of the properties of Bessel functions.

- With weighting function r, $R_m = J_0(\mu_m r)$ are orthogonal functions:

$$\int_0^{r_0} r J_0(\mu_m r) J_0(\mu_k r)\, \mathrm{d}r = 0, \quad \text{for } m \neq k, \tag{3.11a}$$

$$\int_0^{r_0} r J_0^2(\mu_m r)\, \mathrm{d}r = \frac{1}{\mu_m^2} \left[\frac{(\mu_m r)^2}{2} \{ J_0^2(\mu_m r) + J_1^2(\mu_m r) \} \right] \Bigg|_{r=0}^{r=r_0} = \frac{r_0^2}{2} J_1^2(\mu_m r_0), \tag{3.11b}$$

here Eq. (3.7) is inserted.

- Expressing 1 as the sum of $J_0(\mu_m r)$:

$$1 = \sum_{k=1}^{\infty} b_k J_0(\mu_k r). \tag{3.12}$$

Multiplying $r J_0(\mu_m r)$ on both sides and integrating them from 0 to r_0 result in

$$\int_0^{r_0} r J_0(\mu_m r)\, \mathrm{d}r = \sum_{k=1}^{\infty} \int_0^{r_0} b_k r J_0(\mu_k r) J_0(\mu_m r)\, \mathrm{d}r. \tag{3.13}$$

The left side is easily calculated as

$$\int_0^{r_0} r J_0(\mu_m r)\, \mathrm{d}r = \frac{1}{\mu_m^2} [(\mu_m r) J_1(\mu_m r)]|_{r=0}^{r=r_0} = \frac{r_0}{\mu_m} J_1(\mu_m r_0), \tag{3.14a}$$

and the right side from Eqs. (3.11a–b) as

$$= b_m \frac{r_0^2}{2} J_1^2(\mu_m r_0). \tag{3.14b}$$

Hence

$$b_m = \frac{2}{\mu_m r_0 J_1(\mu_m r_0)}. \tag{3.15}$$

3.4. Resultant one-dimensional heat equation in t and x variables

We now start to derive the equations for X_{jm}. Substituting Eq. (3.10) into Eq. (3.2a) and representing 1 as the sum of R_m as Eq. (3.12), Eq. (3.2a) is then re-written as

$$k_j \left(\sum_{m=1}^{\infty} \frac{\mathrm{d}^2 R_m}{\mathrm{d}r^2} X_{jm} + \frac{1}{r} \sum_{m=1}^{\infty} \frac{\mathrm{d}R_m}{\mathrm{d}r} X_{jm} \right) + k_j \sum_{m=1}^{\infty} R_m \frac{\partial^2 X_{jm}}{\partial x^2} = \sum_{m=1}^{\infty} R_m \frac{\partial X_{jm}}{\partial t} + T'_\infty(t) \sum_{m=1}^{\infty} b_m R_m \tag{3.16}$$

where b_m is given in Eq. (3.15).

Inserting Eq. (3.5) results in

$$-\sum_{m=1}^{\infty} k_j \mu_m^2 X_{jm} R_m + k_j \sum_{m=1}^{\infty} \frac{\partial^2 X_{jm}}{\partial x^2} R_m = \sum_{m=1}^{\infty} R_m \frac{\partial X_{jm}}{\partial t} + T'_\infty(t) \sum_{m=1}^{\infty} b_m R_m. \tag{3.17}$$

We finally arrive at

$$k_j \frac{\partial^2 X_{jm}}{\partial x^2} - k_j \mu_m^2 X_{jm} = \frac{\partial X_{jm}}{\partial t} + b_m T'_\infty(t), \quad x \in [L_{j-1}, L_j], \ j = 1, \ldots, n, \tag{3.18a}$$

with the boundary conditions (3.2b)–(3.2f) and the initial condition (3.2g) as

$$-\lambda_1 \frac{\partial X_{1m}}{\partial x}(t, L_0) = -\alpha_+ X_{1m}(t, L_0), \tag{3.18b}$$

$$X_{jm}(t, L_j) = X_{(j+1)m}(t, L_j), \quad j = 1, \ldots, n-1, \tag{3.18c}$$

$$-\lambda_j \frac{\partial X_{jm}}{\partial x}(t, L_j) = -\lambda_{j+1} \frac{\partial X_{(j+1)m}}{\partial x}(t, L_j), \quad j = 1, \ldots, n-1, \tag{3.18d}$$

$$-\lambda_n \frac{\partial X_{nm}}{\partial x}(t, L_n) = \alpha_- X_{nm}(t, L_n), \tag{3.18e}$$

$$X_{jm}(0, x) = -b_m T_\infty(0), \quad x \in [L_{j-1}, L_j], \ j = 1, \ldots, n, \tag{3.18f}$$

where $T_\infty(t) = \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi}$.

The dimension of the equation system is reduced. Let us further simplify the equation by introducing a new variable

$$V_{jm} = X_{jm} + \frac{\mathrm{i}\omega b_m}{k_j \mu_{jm}^2 + \mathrm{i}\omega} T_\infty(t) + \frac{k_j \mu_{jm}^2 b_m}{k_j \mu_{jm}^2 + \mathrm{i}\omega} \mathrm{e}^{-k_j \mu_{jm}^2 t + \mathrm{i}\varphi}. \tag{3.19}$$

Then Eq. (3.18) reads

$$k_j \frac{\partial^2 V_{jm}}{\partial x^2} - k_j \mu_m^2 V_{jm} = \frac{\partial V_{jm}}{\partial t}, \qquad x \in [L_{j-1}, L_j], \ j = 1, \ldots, n, \tag{3.20a}$$

with boundary and initial conditions

$$-\lambda_1 \frac{\partial V_{1m}}{\partial x}(t, L_0) = -\alpha_+ (V_{1m}(t, L_0) - V_+(t)), \tag{3.20b}$$

$$V_{jm}(t, L_j) = V_{(j+1)m}(t, L_j), \quad j = 1, \ldots, n-1, \tag{3.20c}$$

$$-\lambda_j \frac{\partial V_{jm}}{\partial x}(t, L_j) = -\lambda_{j+1} \frac{\partial V_{(j+1)m}}{\partial x}(t, L_j), \quad j = 1, \ldots, n-1, \tag{3.20d}$$

$$-\lambda_n \frac{\partial V_{nm}}{\partial x}(t, L_n) = \alpha_- (V_{nm}(t, L_n) - V_-(t)), \tag{3.20e}$$

$$V_{jm}(0, x) = 0, \quad x \in [L_{j-1}, L_j], \ j = 1, \ldots, n, \tag{3.20f}$$

where

$$V_+(t) = \frac{\mathrm{i}\omega b_m}{k_1 \mu_{1m}^2 + \mathrm{i}\omega} T_\infty(t) + \frac{k_1 \mu_{1m}^2 b_m}{k_1 \mu_{1m}^2 + \mathrm{i}\omega} \mathrm{e}^{-k_1 \mu_{1m}^2 t + \mathrm{i}\varphi}, \tag{3.20g}$$

$$V_-(t) = \frac{\mathrm{i}\omega b_m}{k_n \mu_{nm}^2 + \mathrm{i}\omega} T_\infty(t) + \frac{k_n \mu_{nm}^2 b_m}{k_n \mu_{nm}^2 + \mathrm{i}\omega} \mathrm{e}^{-k_n \mu_{nm}^2 t + \mathrm{i}\varphi}. \tag{3.20h}$$

3.5. Closed form solution

Eq. (3.20) has been studied by Lu et al. [4]. The temporal Laplace transform has been used to the equation. Its definition is given as

$$\overline{V}_{jm}(s, x) = \int_0^\infty \exp(-st) V_{jm}(t, x)\,\mathrm{d}t. \tag{3.21}$$

Without showing the details, we give the closed form solution of V_{jm} as following:

For jth layer, denote

$$q_{jm} = \sqrt{\frac{s}{k_j} + \mu_m^2}, \quad \xi_{jm} = q_{jm} l_j, \qquad h_j = \frac{\lambda_{j+1}}{\lambda_j} \sqrt{\frac{k_j}{k_{j+1}}} \quad (j = 1, \ldots, n-1), \tag{3.22a}$$

$$h_{Am} = \lambda_n q_{nm} \cosh \xi_{nm} + \alpha_- \sinh \xi_{nm}, \qquad h_{Bm} = \lambda_n q_{nm} \sinh \xi_{nm} + \alpha_- \cosh \xi_{nm}, \tag{3.22b}$$

where μ_m is given in Eq. (3.9)

$$\Delta(s) = \begin{vmatrix}
\lambda_1 q_{1m} & -\alpha_+ & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\sinh \xi_{1m} & \cosh \xi_{1m} & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cosh \xi_{1m} & \sinh \xi_{1m} & -h_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \sinh \xi_{2m} & \cosh \xi_{2m} & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh \xi_{2m} & \sinh \xi_{2m} & -h_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \sinh \xi_{(n-1)m} & \cosh \xi_{(n-1)m} & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cosh \xi_{(n-1)m} & \sinh \xi_{(n-1)m} & -h_{n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & h_{Am} & h_{Bm}
\end{vmatrix}, \tag{3.23a}$$

$$\Delta_1(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \qquad \Delta_2(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \tag{3.23b}$$

$$\Delta_3(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \qquad \Delta_4(s) = \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \tag{3.23c}$$

$$M_{jm}(s, x) = -\alpha_+ \Delta_1 \sinh(q_{jm}(x - L_{j-1})) + \alpha_+ \Delta_3 \cosh(q_{jm}(x - L_{j-1})), \tag{3.24a}$$

$$N_{jm}(s, x) = -\alpha_- \Delta_2 \sinh(q_{jm}(x - L_{j-1})) + \alpha_- \Delta_4 \cosh(q_{jm}(x - L_{j-1})). \tag{3.24b}$$

$$\overline{V}_{jm}(s, x) = M_{jm}(s, x) \overline{V}_+(s) + N_{jm}(s, x) \overline{V}_-(s) \tag{3.25}$$

The inverse of $\overline{V}_{jm}(s, x)$ is approximated as

$$\begin{aligned}
V_{jm}(t, x) = {} & \frac{\mathrm{i}\omega b_m}{k_1 \mu_{1m}^2 + \mathrm{i}\omega} M_{jm}(\mathrm{i}\omega, x) \mathrm{e}^{\mathrm{i}\omega t + i\varphi} \\
& + \frac{k_1 \mu_{1m}^2 b_m}{k_1 \mu_{1m}^2 + \mathrm{i}\omega} M_{jm}(-\mu_{1m}^2, x) \mathrm{e}^{-\mu_{1m}^2 t + \mathrm{i}\varphi} \\
& + \frac{\mathrm{i}\omega b_m}{k_n \mu_{nm}^2 + \mathrm{i}\omega} N_{jm}(\mathrm{i}\omega, x) \mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}\varphi} \\
& + \frac{k_n \mu_{nm}^2 b_m}{k_n \mu_{nm}^2 + \mathrm{i}\omega} N_{jm}(-\mu_{nm}^2, x) \mathrm{e}^{-\mu_{nm}^2 t + \mathrm{i}\varphi}.
\end{aligned} \tag{3.26}$$

Combining Eqs. (3.1b), (3.10) and (3.19) gives

$$T_j = \text{real}\left(\sum_{m=1}^{\infty}\left(V_{jm} - \frac{\mathrm{i}\omega b_m}{k_j\mu_{jm}^2 + \mathrm{i}\omega}T_\infty(t) - \frac{k_j\mu_{jm}^2 b_m}{k_j\mu_{jm}^2 + \mathrm{i}\omega}\mathrm{e}^{-k_j\mu_{jm}^2 t + \mathrm{i}\varphi}\right)\times J_0(\mu_m r) + T_\infty(t)\right), \tag{3.27}$$

where μ_m is given in Eq. (3.9), V_{jm} in Eq. (3.26) and 'real' presents the real part.

3.6. More general boundary conditions

For more general time-dependent boundary temperature, we represent it as Fourier approximation as $T_\infty(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(\omega_k t + \varphi_k)$. Due to the linear property of the equation system, the solution is the sum of those with constant boundary temperature a_0 and with the boundary temperature $\sum_{k=1}^{\infty} a_k\cos(\omega_k t + \varphi_k)$. The second part of the solution is easily obtained according to the above-discussed theory.

For the first part of the solution with constant boundary $T_\infty = a_0$, Eq. (3.25) reads (see Eqs. (3.20g) and (3.20h))

$$\overline{V}_{jm}(s,x) = M_{jm}(s,x)\overline{V}_+(s) + N_{jm}(s,x)\overline{V}_-(s) = \left(\frac{\mathrm{i}\omega b_m}{k_1\mu_{1m}^2 + \mathrm{i}\omega}M_{jm} + \frac{\mathrm{i}\omega b_m}{k_n\mu_{nm}^2 + \mathrm{i}\omega}N_{jm}\right)\frac{a_0}{s} + \cdots. \tag{3.28}$$

The omitted term presents the periodic boundary temperature, the inverse of which can be obtained according to the previous-discussed theory.

As the matrix determinant M_{jm} or N_{jm} is the function of hyperbolic functions sinh and cosh which can be approximated by power series, linearisation of Eqs. (3.24) and (3.28) gives

$$\overline{V}_{jm} \approx \frac{\text{const}}{\text{const1} * s + \text{const2}} + \cdots. \tag{3.29}$$

The inverse Laplace transform of the first term is then written as $\frac{\text{const}}{\text{const1}}\exp\left(-\frac{\text{const2}}{\text{const1}}t\right)$. Hence, the final solution can be explicitly obtained.

Another simpler way of finding the first part of the solution with a constant boundary temperature is ignoring the transient term which will die away if studies do not focus very much on the initial temperature change. Then the final value is a_0.

4. Calculation example

A five-layer composite cylinder was selected as the calculation example. Its schematic picture is demonstrated in Fig. 2. The common radius of the cylinders is 1 m. The thermal properties and dimensions of the composition are presented in Table 1. The surface heat transfer coefficients were assumed to be $\alpha_- = 25$ W/m^2/K and $\alpha_+ = 6$ W/m^2/K.

In this example, the boundary temperature was taken from the measurement and then fitted with periodic functions with periods 30, 5, 2 and 1 days as following:

$$T_\infty(t) = a_0 + \sum_{1}^{4} a_i\cos\left(\frac{2\pi t}{\omega_i} - \varphi_i\right) \tag{4.1}$$

where fitting parameters are listed in Table 2 and Fig. 3 shows the values.

Calculations were made at the central points in all cylinders which are marked as layers 1 to 5. Only first nine eigenvalues in Eq. (3.9), taken from some standard textbooks, were used.

The comparison results of transient temperatures in layers 2 and 3 are displayed in Fig. 4. The temperatures were calculated according to seconds in time scale and the results were stored in files as hourly values and shown in figures as hourly and daily values. Results for the first three days are exhibited in Fig. 5. The analytical results agree with the numerical predictions. As the numerical and analytical discrepancies in other layers have not shown any substantial change, we only demonstrate the results in layers 2 and 3 here.

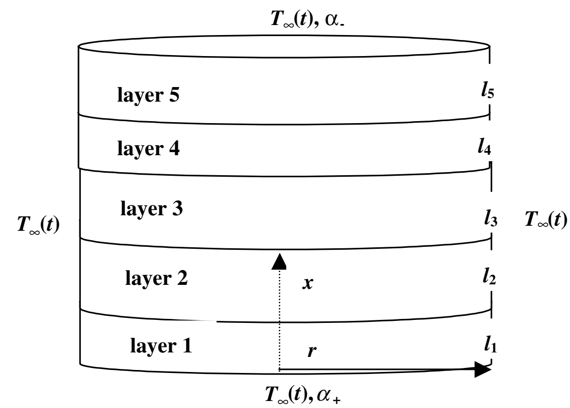

Fig. 2. Schematic picture of the five-layer composite cylinder.

Table 1
Material properties and dimensions of the composite cylinder

Material	Thermal conductivity (W/m/K)	Thermal diffusivity (m^2/s)	Thickness (mm)
Layer 1	0.23	4.11×10^{-7}	50
Layer 2	0.0337	1.47×10^{-6}	100
Layer 3	0.9	3.75×10^{-7}	100
Layer 4	0.147	1.61×10^{-7}	200
Layer 5	0.12	1.5×10^{-7}	20

Table 2
Parameters in Eq. (4.1)

	ω_1	ω_2	ω_3	ω_4
	30.0	5.0	2.0	1.0
	φ_1	φ_2	φ_3	φ_4
	5.607506	13.59596	1.451539	5.418717
a_0	a_1	a_2	a_3	a_4
5.0	2.72217	−5.019664	1.084058	0.4648

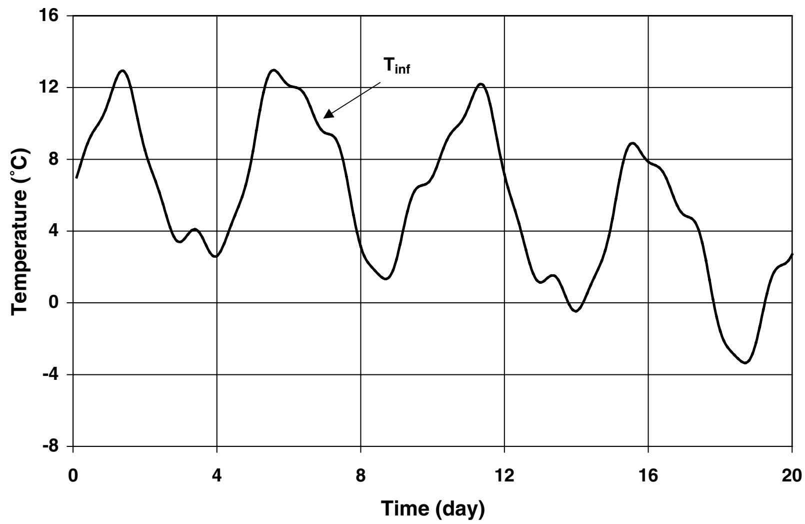

Fig. 3. Time-dependent boundary change.

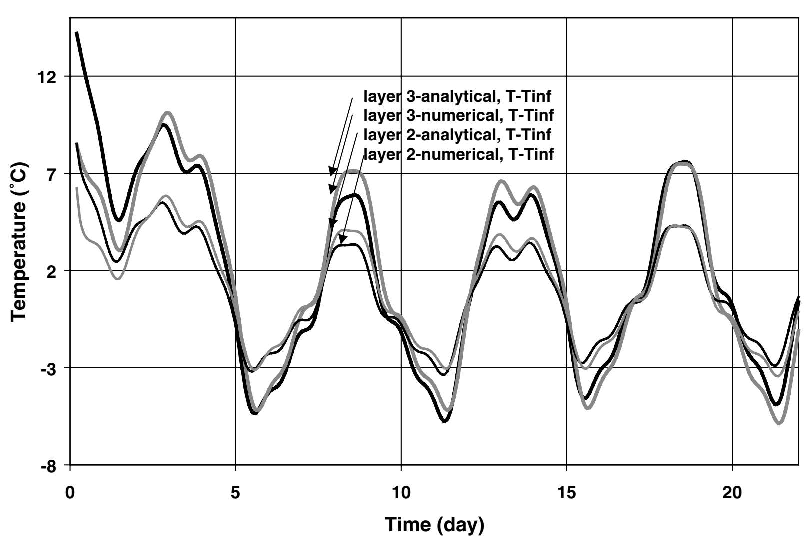

Fig. 4. Comparison of numerical and analytical results in layers 2 and 3.

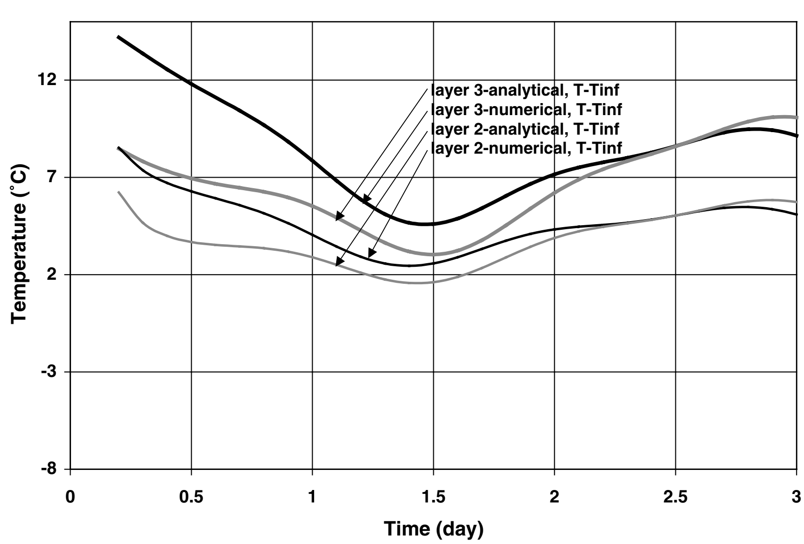

Fig. 5. Comparison of numerical and analytical results in layers 2 and 3.

5. Conclusions

The main conclusion we can draw is the powerful analytical method developed in this paper for the problem of transient heat conduction in multi-dimensional composite cylinder slab with a time-dependent boundary condition. The closed form solution is obtained. Its application range is wide. For such problems, several groups of eigenvalues with imaginary values are needed to compute in most of the published papers. Numerical schemes are then necessitated which may be unstable due to the existence of imaginary eigenvalues. In this paper, however, a rough approximation is sufficient in most cases in the eigenvalue search. Hence, no numerical approach is required. Moreover, in some extreme cases where we have to carry out numerical programs for the eigenvalues, a possible instability of numerical computation due to the imaginary eigenvalue will not exist.

Above all, the computing load is small as calculations involve only simple computations of matrix determinants. Furthermore, it is only for the demonstration sake that we assume a constant conductivity axially and radially in each layer and a perfect contact between layers. These restrictions can be easily cancelled without adding more texts in the article.

Acknowledgement

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